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Chern characters of perfect modules over curved algebras.

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Joint w/ Mark Walker.

k char. 0 field,

A a $\mathbb{Z}$ or $\mathbb{Z}/2$-graded k-algebra.

h a degree 2 element of A.

d a degree 1 k-linear derivation of A s.t.

(i) $d^2(a) = ha - ah$,

(ii) $d(h) = 0$.

(A, d, h) is a curved dg k-algebra.

Ex. M manifold, V v.b. on M.

$$\nabla : V \to V \otimes \Omega^1_M$$

a connection. Then, ∇ extends (uniquely) to a derivation $\tilde{\nabla}$ of $V \otimes \Omega^{\cdot}_M$

$$\nabla^2 = \tilde{\nabla}|_{V \otimes \Omega^{i}_M}.$$

∇ also induces a connection on $End(V)$ given by $[\nabla, -]$.

$$\nabla^2 \in Hom(V, V \otimes \Omega^{\cdot}_M) \simeq \underbrace{End(V) \otimes \Omega^{\cdot}_M}_{\text{some algebra}}.$$

So, $(End(V) \otimes \Omega^{\cdot}_M, [\nabla, -], \nabla^2)$ is a curved dg algebra.

∇^2 is the curvature. It is zero iff the connection is flat.

(Pen ran out of ink.)

From now on, we consider curved dg algebras with $d=0$. So, these are pairs $(A,0,h)$, h a deg. 2 element. These are <u>curved algebras</u>.

$$\Sigma A, \qquad (\Sigma A)^i = A^{i+1}.$$

$$HH(A) := \bigoplus_{n\geq 0} A\otimes(\Sigma A)^{\otimes n} \quad \text{with diff. } b := b_2 + b_0.$$

Denote an element of $A\otimes(\Sigma A)^{\otimes n}$ by $a_0[a_1|\cdots|a_n]$.

$$b_2(a_0[a_1|\cdots|a_n]) = \sum_{i=0}^{n-1} (-1)^{\mathrm{kos}}\, a_0[a_1|\cdots|a_i a_{i+1}|\cdots|a_n]$$
$$+ (-1)^{\mathrm{kos}}\, a_n a_0[a_1|\cdots|a_{n-1}].$$

$$b_0(\qquad\qquad) = \sum_{i=0}^{n} (-1)^{\mathrm{kos}}\, a_0[a_1|\cdots|a_i|h|a_{i+1}|\cdots|a_n].$$

$HH(A)$: Hochschild complex of A.

$HH^{II}(A)$: do products instead of sums.
Hochschild complex of the second kind. (Polishchuk - Positselski, 2013).

<u>Thm</u> (Căldăraru-Tu, 2013). If $h\neq 0$, $HH(A)$ is acyclic.

Def. A mixed complex of k-vs is a dg-Λ-module,
$\Lambda = k[\varepsilon]/(\varepsilon^2)$, $|\varepsilon| = -1$.

$HH^{II}(A)$ (and $HH(A)$) are mixed complexes with Connes B operator.

τ_{n+1} cyclic operator.

s_0 extra degeneracy $\quad a_0[a_1|\cdots|a_n] \longmapsto 1[a_0|a_1|\cdots|a_n]$.

$$B = (1-\tau_{n+2}^{-1})\, s_0 \sum_{r=0}^{n-1} \tau_{n+1}^{r}.$$

$$HN^{II} := (HH^{II}(A)[[u]],\ b+Bu),\ |u| = 2.$$

HKR for curved algebras. Need <u>smoothness</u> for curved algebras.

Def. $A = (A, 0, h)$ is ess. smooth if A is commutative and concentrated in even degrees and the underlying (ungraded) algebra is ess. smooth.

Thm (Efimov, 2013). If A is ess. smooth,

$$HH^{II}(A) \xrightarrow{\sim} (\Omega^{\bullet}_{A/k},\ -\wedge dh,\ d)$$

$a_0[a_1|\cdots|a_n] \longmapsto a_0 da_1 \wedge\cdots\wedge da_n$ $\quad$ diff of the complex. $\quad$ de Rham diff, action of ε

$$\boxed{\begin{array}{l}\textbf{Ex. } A = k[h] \\ A \xrightarrow{\sim} \Omega^1_A \simeq A\cdot dh.\end{array}}$$

Cor. $HN^{II}(A) \simeq (\Omega^{\bullet}_{A/k}[[u]],\ (-\wedge dh + ud))$.

Def. A <u>curved dg cat</u> consists of objects, graded k-v.s. $\mathrm{Hom}(X,Y)$ with deg. 1 endos d.

$\forall X$, $h_X \in \mathrm{End}(X)$ $|h_X| = 2$.

Needs to satisfy

- $d(gf) = d(g)f + (-1)^{|g|} g\, d(f)$.
- $d^2(f) = h_Y f - f h_X$, $f \in \mathrm{Hom}(X,Y)$.
- $d(h_X) = 0$.

Ex. A <u>perfect quasi-A-module</u> is a pair (P, d_P) where P is a graded proj. A-module, d_P deg 1 endo. (P, d_P) is <u>perfect</u> if $d_P^2 = h$.

Ex. - $q\mathrm{Perf}(A)$ curved dg cat.

- $\mathrm{Perf}(A)$ is a dg cat.

There is a notion of HH^{II} of curved dg cats.

Note. $(A, 0)$ is not typically a <u>perfect</u> A-module.

Thm (Morita invariance). The inclusions

$$\mathrm{Perf}(A) \hookrightarrow q\mathrm{Perf}(A) \hookleftarrow A$$

give quasi-isos on HH^{II}.

Def. The chern character of $(P, d_P) \in \mathrm{Perf}(A)$ is the image of $1_P \in HN_0^{\Pi}(\underbrace{\mathrm{End}(P, d_P)}_{\text{dga}})$ under the composition

$$\begin{tikzcd} HN_0^{II}(\mathrm{End}(P, d_P)) \arrow[r] & HN_0^{II}(q\mathrm{Perf}(A)) \arrow[d, "\wr"] \\ & HN_0^{II}(A) \arrow[d, "HKR"] \\ & H_0(\Omega^\bullet_{A/k}[[u]], (dh \wedge -) + ud). \end{tikzcd}$$

Remark. Trace is not S^1-equivariant.
Or, does not commute with B.
What does this say about S^1-structures?

Def. A connection on (P, d_P) is just a connection on P, in the classical sense. Given (P, d_P), equip it w/ a connection ∇. The curvature of ∇ is $\nabla^2 \underset{\uparrow}{u} + [\nabla, d_P] \in \mathrm{End}_A(P) \otimes \Omega^\bullet_{A/k}[[u]]$.
formal var.

There is a trace

$$tr : \mathrm{End}_A(P) \otimes \Omega^\bullet_{A/k}[[u]] \longrightarrow \Omega^\bullet_{A/k}[[u]].$$

Thm (Brown–Walker). $$ch(P, d_P) = tr\left(e^{\overbrace{(\nabla^2 u + [\nabla, d_P])}^{R}}\right) = tr\left(1 + R + \frac{R^2}{2} + \dots\right).$$

Motivation for looking at $[\nabla, d_P]$ comes from Quillen's work on super-connections.

Ex. $A = k[x_1, \dots, x_n]$, h s.t. A/h has only isolated sings. $\mathbb{Z}/2$-graded case.

$$HH_*^{II}(A) \cong H_*(\Omega^\bullet_{A/k}, -dh) \cong \frac{k[x_1, \dots, x_n]}{(\frac{\partial h}{\partial x_i})}[n] \quad \text{Jacobian algebra.}$$